1— $\vec{v} = (1, 3-i, 2+3i)$

$\vec{w} = (-4-4i, 1+2i, 3-i)$

$(\vec{v}, \vec{v}) = (1)^2 + (3-i)^*(3-i) + (2+3i)^*(2+3i)$

$= 1 + (9+1) + (4+9) = 1 + 10 + 13 = 24$

$(\vec{w}, \vec{w}) = (-4-4i)^*(-4-4i) + (1+2i)^*(1+2i) + (3-i)^*(3-i)$

$= (16+16) + (1+4) + (9+1) = 32 + 5 + 10 = 47$

$(\vec{w}, \vec{v}) = (-4-4i)^*(1) + (1+2i)^*(3-i) + (3-i)^*(2+3i)$

$= -4+4i + (3-6i-i-2) + (6+9i+2i-3)$

$= (-4+1+3) + i(4-6-1+9+2) = i(8)$

$(\vec{v}, \vec{w}) = (\vec{w}, \vec{v})^* = -8i$

$|(\vec{v}, \vec{w})|^2 = 64, \quad ||\vec{v}||^2 = 24, \quad ||w||^2 = 47$

$64 < (24)(47)$

2- $(0, 1, x) = \vec{v}_1$

$(x, 0, 1) = \vec{v}_2$

$(x, 1, 1+x) = \vec{v}_3$

For $\vec{v}_1, \vec{v}_2, \vec{v}_3$ to be independent we must have

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$$

or $$c_1(0, 1, x) + c_2(x, 0, 1) + c_3(x, 1, 1+x) = 0$$

$$\Rightarrow \left((c_2+c_3)x,\ c_1+c_3,\ (c_1+c_3)x + c_2\right) = 0$$

or $(c_2+c_3)x = 0$ (1), $c_1 + c_3 = 0$ (2), $(c_1+c_3)x + c_2 = 0$ (3)

(2) in (3) $\rightarrow$ $c_2 = 0$ in (1) $\rightarrow$ $c_3 x = 0$

From (2) $\rightarrow$ $c_3 = -c_1$ $\therefore \begin{cases} c_3 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases}$ if $\underline{\underline{x \neq 0}}$ ✓

3 - a) $T\,p(x) = p(x^2)$

let $p(x)$, $q(x)$ be 2 different functions

$T(\alpha p(x)) = \alpha p(x^2)$

$T(\alpha p(x) + \beta q(x)) = \alpha p(x^2) + \beta q(x^2) = \alpha T(p(x)) + \beta T(q(x)) \rightarrow T$ linear

b) $T(\alpha p(x)) = (\alpha p(x))^2 = \alpha^2 (p(x))^2 \rightarrow T$ is not linear

c) $T(\alpha p(x)) = x^2(\alpha p(x)) = \alpha x^2 p(x) = \alpha T(p(x))$

$T(\alpha p(x) + \beta q(x)) = x^2(\alpha p(x) + \beta q(x)) = \alpha T(p(x)) + \beta T(q(x))$

$\rightarrow$ linear

4. $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

Eigenvalues: $|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 6 = 0 \rightarrow -4 - 3\lambda + \lambda^2 = 0$

$(\lambda - 4)(\lambda + 1) = 0 \rightarrow \lambda = -1$ or $\lambda = 4$

$\lambda_1 = -1, \quad \lambda_2 = 4$

Eigenvectors $\begin{pmatrix} 1-\lambda_1 & 2 \\ 3 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \rightarrow$ $(1-\lambda_1)a + 2b = 0$ or $2a + 2b = 0 \rightarrow b = -a$

$\therefore \vec{V}_1 = \begin{pmatrix} a \\ -a \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, normalize $(V_1, V_1) = 1 \Rightarrow 2a^2 = 1$

$a = \frac{1}{\sqrt{2}} \quad \therefore \vec{V}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(1-\lambda_2)a + 2b = 0 \rightarrow -3a + 2b = 0 \Rightarrow b = \frac{3}{2}a$

$\vec{V}_2 = \begin{pmatrix} a \\ \frac{3}{2}a \end{pmatrix} = a\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$, $(\vec{V}_2, \vec{V}_2) = 1 = a^2(1 + \frac{9}{4}) = \frac{13}{4}a^2$

$a = \frac{2}{\sqrt{13}}, \quad \vec{V}_2 = \frac{2}{\sqrt{13}}\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

$\therefore P = (\vec{V}_1, \vec{V}_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{13}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{13}} \end{pmatrix}$ (Note you do not have to normalize $\vec{V}_1$ & $\vec{V}_2$)

5. $\vec{x}' = R\vec{x}$ $\quad (\vec{x}', \vec{x}') = (R\vec{x}, R\vec{x}) = (R^{\dagger}R\vec{x}, \vec{x}) = (\vec{x}, \vec{x})$

$\therefore R^{\dagger}R = 1$ but R is real, $R^{\dagger} = R^{T}$

$\therefore \boxed{R^{T}R = I}$ $\quad R$ is orthogonal matrix

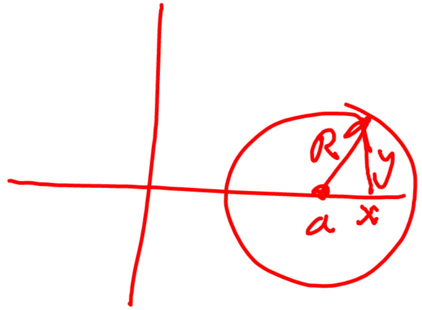
6. $\vec{F} = (x+y^2)\vec{i} + (xy-1)\vec{j}$

$\oint \vec{F}\cdot d\vec{r} = \oint_C (x+y^2)\,dx + (xy-1)\,dy$

a) C is a circle with center on x-axis : Center $= (a, 0)$

Let R be radius of circle : $x - a = R\cos\theta$

$y = R\sin\theta$



$\therefore dx = -R\sin\theta\, d\theta$

$dy = R\cos\theta\, d\theta$

$(x+y^2)\,dx = (a + R\cos\theta +, R^2\sin^2\theta)(-R\sin\theta\, d\theta)$

$(xy-1)\,dy = (a + R\cos\theta)(R\sin\theta)(R\cos\theta\, d\theta)$

We $\int_0^{2\pi}\sin\theta\, d\theta = 0 = \int_0^{2\pi}\cos\theta\, d\theta$

$\int_0^{2\pi}\sin\theta\cos\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}\sin 2\theta\, d\theta = 0$

$$\int_0^{2\pi}\sin^3\theta\,d\theta = \int_0^{2\pi}\sin\theta\,(1-\cos^2\theta)\,d\theta = -\int_0^{2\pi}\sin\theta\cos^2\theta\,d\theta$$

$$= +\int_0^{2\pi}\cos^2\theta\,d(\cos\theta) = \frac{1}{3}\cos^3\theta\Big|_0^{2\pi} = 0$$

All integrals of periodic functions are zero

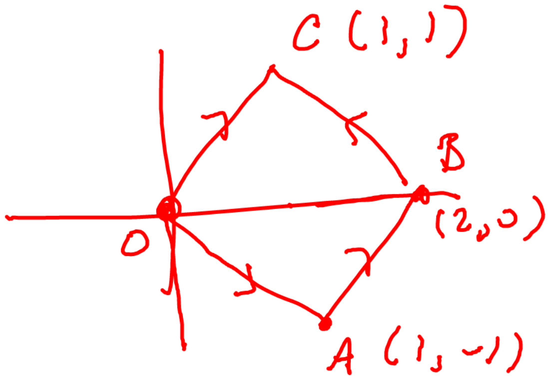
$$\Rightarrow \quad \oint \vec F\cdot d\vec r = 0$$

$$\nabla\times\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & xy-1 & 0 \end{vmatrix} = \vec i\,(0) - \vec j\,(0) + \vec k\left(\frac{\partial}{\partial x}(xy-1) - \frac{\partial}{\partial y}(x+y^2)\right)$$

$\vec F$ not conservative

$$= \vec k\,(y-2y) = -\vec k\,(y)$$

b) on a square



on $\underline{OA}$ $\quad y=-x$

AB $\quad y = x-2$

BC $\quad y = -x+2$

on $\underline{OA}$ $\quad \vec F\cdot d\vec r = (x+y^2)\,dx + (xy-1)\,dy = (x+x^2)\,dx - (-x^2-1)\,dx$

$$= (x+x^2+x^2+1) = (x+2x^2+1)\,dx$$

on ∂A $\int_{\partial A} \vec{F}\cdot d\vec{r} = \int_0^1 (x+2x^2+1)\,dx = \frac{x^2}{2}+\frac{2}{3}x^3+x\Big|_0^1 = \frac{1}{2}+\frac{2}{3}+1$

on AB $\rightarrow$ $\int_1^2 \left(x+(x-2)^2+x(x-2)-1\right)dx = \int_1^2 \left(x+x^2-4x+4+x^2-2x-1\right)dx$

$= \int_1^2 (2x^2-5x+3)\,dx = \frac{2}{3}x^3-\frac{5}{2}x^2+3x\Big|_1^2$

$= \frac{2}{3}(8-1)-\frac{5}{2}(4-1)+3(2-1) = \frac{14}{3}-\frac{15}{2}+3$

on BC $\rightarrow$ $\int_2^1 \left(x+(2-x)^2\right)dx+\left(x(2-x)-1\right)(-dx)$

$= \int_2^1 \left(x+4-4x+x^2-2x+x^2+1\right)dx$

$= \int_2^1 (2x^2-\cancel{8}5x+5)\,dx = \frac{2}{3}x^3-\frac{5}{2}x^2+5x\Big|_2^1$

$= \frac{2}{3}(1-8)-\frac{5}{2}(1-4)+5(1-2) = -\frac{14}{3}+\frac{15}{2}-5$

on $\underline{CO}$ $y=x$ $\rightarrow$ $\int_1^0 (x+x^2+x^2-1)\,dx = \int_1^0 (2x^2+x-1)\,dx$

$= \frac{2}{3}x^3+\frac{x^2}{2}-x\Big|_1^0 = -\frac{2}{3}-\frac{1}{2}+1$

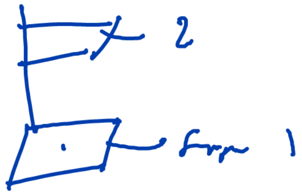
$\therefore \oint = \frac{1}{2}+\frac{2}{3}+1+\frac{14}{3}-\frac{15}{2}+3-\frac{14}{3}+\frac{15}{2}-5-\frac{2}{3}-\frac{1}{2}+1$

$= \frac{1}{2}(1-15+3-1)+\frac{1}{3}(2+14-14-2)+(1+3-5+1)$

$= -6$

Not ~~conservative~~

7. $\int_S \vec{F}\cdot d\vec{a}$ $\quad \vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$



on Face 1: $z = 0$ $\quad d\vec{a} = dx\,dy\,(-\vec{k})$

$\vec{F}\cdot d\vec{a} = -z^2\,dx\,dy = 0$

on Face 2 $\quad z = 1$ $\quad d\vec{a} = dx\,dy\,(\vec{k})$

$\vec{F}\cdot d\vec{a} = z^2\,dx\,dy = dx\,dy$

$\longrightarrow \int_S dx\,dy = 1$ (area of square)

Expression is symmetric with respect to x, y, z

$\therefore$ Face 3 + Face 4 = Face 5 + Face 6 = Face 1 + Face 2

$\longrightarrow \int_S \vec{F}\cdot d\vec{a} = 3 = (1+1+1)$

8. $\nabla\cdot\vec{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}(F_\varphi) + \frac{\partial}{\partial z}F_z$

$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot\rho^3) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}(\rho^2\sin\varphi) + \frac{\partial}{\partial z}(z^2)$

$= 4\rho^2 + \rho\cos\varphi + 2z$

9. $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) \stackrel{?}{=} \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 \vec{v}$

Do it in cartesian: let $\vec{\nabla} \times \vec{v} = \vec{c}$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$c_3 = (\vec{\nabla} \times \vec{v})_3 = \partial_1 v_2 - \partial_2 v_1$$

$$c_1 = (\vec{\nabla} \times \vec{v})_1 = \partial_2 v_3 - \partial_3 v_2$$

$$c_2 = (\vec{\nabla} \times \vec{v})_2 = \partial_3 v_1 - \partial_1 v_3$$

$$(\vec{\nabla} \times \vec{c})_1 = \partial_2 c_3 - \partial_3 c_2 = \partial_2(\partial_1 v_2 - \partial_2 v_1) - \partial_3(\partial_3 v_1 - \partial_1 v_3)$$

$$= \partial_1(\partial_2 v_2) - \partial_2^2 v_1 - \partial_3^2 v_1 + \partial_1(\partial_3 v_3)$$

$$= \partial_1(\partial_2 v_2 + \partial_3 v_3 + \partial_1 v_1) - (\partial_1^2 v_1 + \partial_2^2 v_1 + \partial_3^2 v_1)$$

$$(\vec{\nabla} \times (\vec{\nabla} \times \vec{v}))_1 = \partial_1(\vec{\nabla} \cdot \vec{v}) - \nabla^2 v_1$$

$\therefore$ this is true for all other components

10. problem 10 not included

$$\int_{-\infty}^{\infty} f(x)\, \delta_n(x-a)\, dx = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x) \cdot n\, dx$$

$$= n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \left(f(0) + x f'(0) + \cdots \right) dx = f(0) + f'(0) \cdot n \frac{x^2}{2} \Big|_{-\frac{1}{2n}}^{\frac{1}{2n}} + O\left(\frac{1}{n^2}\right) \to 0$$